ONE-DIMENSIONAL RINGS OF FINITE F-REPRESENTATION TYPE

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Abstract. We prove that a complete local or graded one-dimensional domain of prime characteristic has finite F-representation type if its residue field is algebraically closed or finite, and present examples of a complete local or graded one-dimensional domain which does not have finite F-representation type with a perfect residue field. We also present some examples of higher dimensional rings of finite F-representation type.

1. Introduction

Smith and Van den Bergh [2] introduced the notion of finite F-representation type as a characteristic p analogue of the notion of finite Cohen-Macaulay representation type. Rings of finite F-representation type satisfy several nice properties. For example, Seibert [1] proved that the Hilbert-Kunz multiplicities are rational numbers, Yao [4] proved that tight closure commutes with localization in such rings, and Takagi–Takahashi [3] proved that if R is a Cohen-Macaulay ring of finite F-representation type with canonical module ω_R, then $H^n_I(\omega_R)$ has only finitely many associated primes for any ideal I of R and any integer n. However, it is difficult to determine whether a given ring have finite F-representation type. In this paper, we prove the following results: Let A be a complete or graded one-dimensional domain of prime characteristic with the residue field k. Then

(1) if k is algebraically closed, then any finitely generated A-module has finite F-representation type,
(2) if k is finite, then A has finite F-representation type,
(3) there exist examples of rings which do not have finite F-representation type with k perfect.

In the last section, we give some examples of finite F-representation type of dimension higher than one. There is a question posed by Brenner:

Question 1.1 (Brenner)**.** Let k be an algebraically closed field of characteristic p. Then does the ring $k[x, y, z]/(x^2 + y^3 + z^7)$ have finite F-representation type?

We prove that $k[x, y, z]/(x^2 + y^3 + z^7)$ has finite F-representation type if $p = 2, 3$ or 7.

2. Rings of finite F-representation type

Throughout this paper, all rings are Noetherian commutative rings of prime characteristic p. We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of non-negative integers. If

$R = \bigoplus_{i\geq 0} R_i$ is an $\mathbb{N}$-graded ring, we assume that $\gcd\{i \mid R_i \neq 0\} = 1$. We denote by $R_+ = \bigoplus_{i>0} R_i$ the unique homogeneous maximal ideal of R.

The Frobenius map $F : R \to R$ is defined by sending r to r^p for all $r \in R$. For any R-module M, we denote by ${}^eM$ the module M with its R-module structure pulled back via the e-times iterated Frobenius map $F^e : r \mapsto r^{p^e}$, that is, ${}^eM$ is the same as M as an abelian group, but its R-module structure is determined by $r \cdot m := r^{p^e}m$ for $r \in R$ and $m \in M$. If ${}^1R$ is a finitely generated R-module (or equivalently ${}^eR$ is a finitely generated R-module for every $e \geq 0$), we say that R is F-finite. In general, if ${}^1M$ is a finitely generated R-module, we say that M is F-finite. Remark that when R is reduced, ${}^eR$ is isomorphic to $R^{1/q}$ where $q = p^e$. If R and M are $\mathbb{Z}$-graded, then ${}^eM$ carries a $\mathbb{Q}$-graded R-module structure: We grade ${}^eM$ by putting $[{}^eM]_\alpha = [M]_{p^e\alpha}$ if $\alpha \in \frac{1}{p^e}\mathbb{Z}$, otherwise $[M]_\alpha = 0$. For $\mathbb{Q}$-graded modules M and N and a rational number r, we say that a homomorphism $\phi : M \to N$ is *homogeneous (of degree r)* if $\psi(M_s) \subset N_{s+r}$ for all $s \in \mathbb{Q}$. We denote by $\mathrm{Hom}_r(M, N)$ the group of homogeneous homomorphisms of degree r, and set $\underline{\mathrm{Hom}}_R(M, N) = \bigoplus_{r\in\mathbb{Q}} \mathrm{Hom}_r(M, N)$.

Let I be an ideal of R. Then for any $q = p^e$, we use $I^{[q]}$ to denote the ideal generated by $\{x^q \mid x \in I\}$. For any R-module M, it is easy to see that $(R/I)\otimes_R {}^eM \cong {}^eM/(I \cdot {}^eM) \cong {}^e(M/I^{[q]}M)$. Since the functor ${}^e(-)$ is a exact functor, and I and $I^{[q]}$ have the same radical, we have ${}^eH_I(M) \cong {}^eH_{I^{[q]}}(M) \cong H_I({}^eM)$.

Definition 2.1.

(1) Let R be a ring of prime characteristic p, and M a finitely generated R-module. We say that M has *finite F-representation type* by finitely generated R-modules $M_1, \cdots, M_s$ if for every $e \in \mathbb{N}$, the R-module ${}^eM$ is isomorphic to a finite direct sum of the R-modules $M_1, \cdots, M_s$, that is, there exist nonnegative integers $n_{e,1}, \cdots, n_{e,s}$ such that

$$ {}^eM \cong \bigoplus_{i=1}^{s} M_i^{\oplus n_{e,i}}. $$

We say that a ring R has finite F-representation type if R has finite F-representation type as an R-module.

(2) Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian graded ring of prime characteristic p, and M a finitely generated graded R-module. We say that M has *finite graded F-representation type* by finitely generated $\mathbb{Q}$-graded R-modules $M_1, \ldots, M_s$ if for every $e \in \mathbb{N}$, the $\mathbb{Q}$-graded R-module ${}^eM$ is isomorphic to a finite direct sum of the $\mathbb{Q}$-graded R-modules $M_1, \ldots, M_s$ up to shift of grading, that is, there exist non-negative integers n_{ei} for $1 \leq i \leq s$, and rational numbers $a_{ij}^{(e)}$ for $1 \leq j \leq n_{ei}$ such that there exists a $\mathbb{Q}$-homogeneous isomorphism

$$ {}^eM \cong \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{n_{ei}} M_i(a_{ij}^{(e)}), $$

where $M(a)$ stands for the module obtained from $\mathbb{Q}$-graded module M by the shift of grading by $a \in \mathbb{Q}$; $[M(a)]_b := M_{a+b}$ for $b \in \mathbb{Q}$. We say that a

graded ring R has finite graded F-representation type if R has finite graded F-representation type as a graded R-module.

Note that if M has finite F-representation type, then M is F-finite. In this paper, we mainly investigate the cases where R is a complete local Noetherian ring or $\mathbb{N}$-graded ring $R = \bigoplus_{i \geq 0} R_i$ with $R_0 = k$ a field. Remark that the Krull-Schmidt theorem holds in these cases.

Example 2.2.

(i) Direct sums, localizations, or completions of modules of finite F-representation type also have finite F-representation type.

(ii) Let R be an F-finite regular local ring or a polynomial ring $k[t_1, \dots, t_r]$ over a field k of characteristic $p > 0$ such that $[k : k^p] < \infty$. Then R has finite F-representation type.

(iii) Let R be Cohen-Macaulay local (resp. graded) ring with finite (resp. graded) Cohen-Macaulay representation type, that is, there are finitely many isomorphism classes of indecomposable (resp. graded) maximal Cohen-Macaulay R-modules. Then every finitely generated (resp. graded) maximal Cohen-Macaulay R-modules has finite F-representation type.

(iv) Let $(R, \mathfrak{m}, k)$ be an F-finite local ring (resp. $\mathbb{N}$-graded ring with $k = R/R_+ \cong R_0$), and M an R-module of finite length $\ell(M)$. Then M has finite F-representation type; ${}^eM \cong k^{\ell(M)a^e}$ for sufficiently large $q = p^e$ where $a = [k : k^p]$. In particular, Artinian F-finite local rings have finite F-representation type.

(v) Let $R \hookrightarrow S$ be a finite local homomorphism of Noetherian local rings of prime characteristic p such that R is an R-module direct summand of S. If S has finite F-representation type, so does R.

(vi) ([2], Proposition 3.1.6]) Let $R = \bigoplus_{i \geq 0} R_i \subset S = \bigoplus_{i \geq 0} S_i$ be a Noetherian $\mathbb{N}$-graded rings with R_0 and S_0 fields of characteristic $p > 0$ such that R is an R-module direct summand of S. Assume in addition that $[S_0 : R_0] < \infty$. If S has finite graded F-representation type, so does R. In particular, normal semigroup rings and rings of invariants of linearly reductive groups have finite graded F-representation type.

3. One-dimensional case

In this section, we investigate whether one-dimensional complete local or $\mathbb{N}$-graded domains have finite F-representation type.

Theorem 3.1. *Let $(A, \mathfrak{m}, k)$ be a one-dimensional complete local domain (resp. an $\mathbb{N}$-graded domain $A = \bigoplus_{i \geq 0} A_i$, $\mathfrak{m} = \bigoplus_{i > 0} A_i$ and $A_0 \cong A/\mathfrak{m} = k$) of prime characteristic p. Let M be a finitely generated (resp. graded) A-module. Assume that k is an algebraically closed field. Then for sufficiently large $e \gg 0$,*

$$ {}^eM \cong B^{\oplus rq} \oplus k^{\ell} \quad (q = p^e) $$

where B is the integral closure of A, r is the rank of M, and ℓ is the length of $H^0_{\mathfrak{m}}(M)$. In particular, M has finite F-representation type.

Proof. In the case where A is a complete local domain, B is isomorphic to a formal power series ring $k[[t]]$. For $f \in B$, we set $v_B(f) = \min\{i \mid f \in t^iB\}$. Let $H = \{v_B(f) \mid f \in A\}$, and $c(H) = \min\{j \mid i \in H \text{ if } i \geq j\}$. Since $\mathbb{N}\backslash H$ is a finite set and A is complete, it follows that $t^i \in A$ for all $i \geq c(H)$. Let $n = \min\{n \mid \mathfrak{m}^n H^0_{\mathfrak{m}}(M) = 0\}$ and $q = p^e \geq \max\{c(H), n\}$. Since $B^q \subset A$, ${}^eM$ has a B-module structure. Thus ${}^eM \cong B^{\oplus rq} \oplus H^0_{\mathfrak{m}}({}^eM)$ because B is a principal ideal domain and $\mathrm{rank}({}^eM) = rq$. Since $H^0_{\mathfrak{m}}({}^eM) \cong^e H^0_{\mathfrak{m}}(M) \cong k^{\ell}$, we conclude the assertion.

In the case where A is an $\mathbb{N}$-graded ring, B is isomorphic to a polynomial ring $k[t]$. Since $A = k[t^{n_1}, \ldots, t^{n_r}]$ for some $n_i \in \mathbb{N}$ with $\gcd(n_1, \ldots, n_r) = 1$, we can prove the assertion similarly to the complete case. □

The assumption that k is algebraic closed is essential for this theorem. Let $A = \bigoplus_{i\geq 0} A_i$ be a one-dimensional $\mathbb{N}$-graded domain with $A_0 = k$ a perfect field. Then ${}^eA \cong A^{1/q}$ has rank $[k : k^q]q = q$, and is decomposed to A-modules of rank one by degree; $A^{1/q} = \bigoplus_{i=0}^{q-1} M_i^{(e)}$ where

$$M_i^{(e)} = \bigoplus_{j\equiv i \mod q} [A^{\frac{1}{q}}]_{\frac{j}{q}}$$

where $[A^{\frac{1}{q}}]_{\frac{j}{q}}$ is the degree $\frac{j}{q}$ component of $A^{\frac{1}{q}}$. Let B be the integral closure of A. Then it follows that B is isomorphic to a polynomial ring $K[t]$ with $\deg t = 1$ for some K, a finite degree extension of k. Note that K is also a perfect field. We can write $A = k[\alpha_1 t^{n_1}, \ldots, \alpha_r t^{n_r}]$ for some $n_1, \ldots, n_r \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_r \in K$. For $i \in \mathbb{N}$, we define

$$V_i := \{\alpha \in K \mid \alpha t^i \in A\}$$

the k-vector subspace of K which is a coefficient of t^i in A. We have $V_i = K$ for all sufficiently large i because B/A is a graded A-module of finite length. We set

$$c = \min\{i \mid V_j = K \text{ for all } j \geq i\}.$$

For $q = p^e \geq c$, we have

$$M_i^{(e)} = \begin{cases} \bigoplus_{j\geq 1} K \cdot t^{j+\frac{i}{q}} & (V_i = 0), \\ \bigoplus_{j\geq 0} K \cdot t^{j+\frac{i}{q}} & (V_i = K), \\ V_i^{\frac{1}{q}} \cdot t^{\frac{i}{q}} \oplus (\bigoplus_{j\geq 1} K \cdot t^{j+\frac{i}{q}}) & (0 \subsetneq V_i \subsetneq K). \end{cases}$$

It is easy to see that $M_i^{(e)} \cong B$ if $V_1 = 0$ or K. Note that $V_i^{1/q} = \{\alpha^{1/q} \mid \alpha \in V_i\}$ is also a k-vector subspace of K since K is a perfect field.

Lemma 3.2. *Let the notation be as above. Let* $q_1 = p^{e_1}$, $q_2 = p^{e_2} \geq c$ *and* $i_1, i_2 \geq 0$ *such that* $0 \subsetneq V_{i_1}, V_{i_2} \subsetneq K$. *Then* $M_{i_1}^{(e_1)}$ *is isomorphic to* $M_{i_2}^{(e_2)}$ *as graded module up to shift of grading if and only if* $\beta V_{i_1}^{1/q_1} = V_{i_2}^{1/q_2}$ *for some* $\beta \in K^* = K\setminus\{0\}$.

Proof. A graded homomorphism $\phi : M_{i_1}^{(e_1)} \to M_{i_2}^{(e_2)}$ can be identified with some homogeneous element of B:

$$\underline{\mathrm{Hom}}_A(M_{i_1}^{(e_1)}, M_{i_2}^{(e_2)}) \hookrightarrow \underline{\mathrm{Hom}}_C(C \cdot t^{\frac{i_1}{q_1}}, C \cdot t^{\frac{i_2}{q_2}}) \cong C\Big(\frac{i_1}{q_1} - \frac{i_2}{q_2}\Big),$$

where $C = B[t^{-1}] = K[t, t^{-1}]$. Let $\phi \in \mathrm{Hom}_A(M_{i_1}^{(e_1)}, M_{i_2}^{(e_2)})$ be a homogeneous homomorphism which maps to a homogeneous element $\beta t^n \in C\Big(\frac{i_1}{q_1} - \frac{i_2}{q_2}\Big)$ under the above inclusion. Then for $g{\cdot}t^{i_1/q} \in M_{i_1}^{(e_1)} \subset Bt^{i_1/q}$ with $g \in B$, $\phi(g{\cdot}t^{i_1/q}) = \beta g{\cdot}t^{n+i_2/q}$. Hence $n \geq 0$ and ϕ is an isomorphism if and only if $n = 0$ and $\beta V_{i_1}^{1/q_1} = V_{i_2}^{1/q_2}$. Hence there is one-to-one correspondence between the set of graded isomorphisms from $M_{i_1}^{(e_1)}$ to $M_{i_2}^{(e_2)}$ and the set $\{\beta \in K^* \mid \beta V_{i_1}^{1/q_1} = V_{i_2}^{1/q_2}\}$.

□

We will present examples of one-dimensional domain which does not have finite (graded) F-representation type.

Example 3.3. Let $k = \bigcup_{e \geq 1} \mathbb{F}_2(u^{1/2^e})$ be the perfect closure of a rational function field $\mathbb{F}_2(u)$. Let $A = k[x,y]/(x^4 + x^2y^2 + uxy^3 + y^4)$, $\deg x = \deg y = 1$, and $\hat{A} = k[[x,y]]/(x^4 + x^2y^2 + uxy^3 + y^4)$. Since $x^4 + x^2 + v^q x + 1$ is a reduced polynomial in $\mathbb{F}_2[v, x]$ for all $q = 2^e$, it follows that $x^4 + x^2y^2 + uxy^3 + y^4$ is a reduced polynomial in $k[x,y]$. We will prove that A does not have finite graded F-representation type, and $\hat{A}$ does not have finite F-representation type.

Let $\alpha \in \overline{k}$ be a root of the irreducible polynomial $x^4 + x^2 + ux + 1$, and set $K = k(\alpha)$. Then $A \cong k[\alpha t, t] \subset K[t]$ and the integral closure B of A is isomorphic to $K[t]$, a polynomial ring over K. Note that $0 \subsetneq V_i \subsetneq K$ if and only if $0 \leq i \leq 2$, and $V_i = \bigoplus_{j=0}^{i} k\alpha^j$ for $0 \leq i \leq 2$ and $V_i = K$ for all $i \geq 3$.

We will show that $M_1^{(e_2)} \ncong M_1^{(e_1)}$ for any $e_2 > e_1 \geq 2$. Assume, to the contrary, that $M_1^{(e_2)} \cong M_1^{(e_1)}$ for some $e_2 > e_1 \geq 2$. We set $q_i = 2^{e_i}$, and $e = e_2 - e_1$. Then there exists $\beta \in K^*$ such that $\beta V_1^{1/q_1} = V_1^{1/q_2}$ by the previous lemma. Since $V_1^{1/q} = k \oplus k\alpha^{1/q}$, there exist $a, b, c, d \in k$ such that

$$\beta = a + b\alpha^{1/q_2},\ \beta\alpha^{1/q_1} = c + d\alpha^{1/q_2}.$$

It follows that

$$b^{q_2}\alpha^{2^e+1} + a^{q_2}\alpha^{2^e} - d^{q_2}\alpha - c^{q_2} = 0.$$

We will show that $1, \alpha, \alpha^{2^e}, \alpha^{2^e+1}$ are linearly independent over k for any $e \geq 1$. If this is proved, then $a = b = c = d = 0$ which contradicts that $\beta \neq 0$. We claim that or $e \geq 2$ there exist polynomials $f_e, g_e, h_e \in k[u]$ such that $f_e \neq 0$, $g_e \neq 0$,

$$\alpha^{2^e} = f_e\alpha^2 + g_e\alpha + h_e$$

and $\deg_u f_e = \deg_u g_e - 1$ if e is even and $\deg_u f_e = \deg_u g_e + 1$ if e is odd. We prove this claim by induction on e. If $e = 2$, then $\alpha^4 = \alpha^2 + u\alpha + 1$ and thus $f_2 = 1$ and $g_2 = u$. If the claim holds true for e, then

$$\alpha^{2^{e+1}} = f_e^2\alpha^4+g_e^2\alpha^2+h_e^2 = f_e^2(\alpha^2+u\alpha+1)+g_e^2\alpha^2+h_e^2 = (f_e^2+g_e^2)\alpha^2+uf_e^2\alpha+f_e^2+h_e^2,$$

and thus $f_{e+1} = f_e^2 + g_e^2$ and $g_{e+1} = uf_e^2$. Note that $f_{e+1} \neq 0$ as $\deg_u f_e \neq \deg_u g_e$. Since $\deg_u f_{e+1} = 2\max\{\deg_u f_e, \deg_u g_e\}$ and $\deg g_{e+1} = 2\deg_u f_e + 1$, the claim also holds true for $e+1$. By induction, the claim is true for every $e \geq 2$. Hence it follows that $1, \alpha, \alpha^{2^e}, \alpha^{2^e+1}$ are linearly independent over k for all $e \geq 1$ as $1,\ \alpha,\ \alpha^2,\ \alpha^3$ are linearly independent over k. Therefore $M_1^{(e_2)} \not\cong M_1^{(e_1)}$ for all $e_2 > e_1 \geq 3$. Hence A does not have finite graded F-representation type.

We will prove that $\hat{A}$ does not have finite F-representation type. Let $\hat{B}$ be the integral closure of $\hat{A}$. Note that $B \cong B \otimes_A \hat{A} = K[[t]]$ and $\hat{A}^{1/q} \cong \bigoplus_{i=0}^{q-1} \hat{M}_i^{(e)}$ where $\hat{M}_i^{(e)} = M_i^{(e)} \otimes \hat{A}$. Assume that there is an isomorphism $\phi : \hat{M}_1^{(e_2)} \to \hat{M}_1^{(e_1)}$ for some $e_2 > e_1 \geq 3$. Let $\hat{\Psi}$ be the inclusion $\operatorname{Hom}_{\hat{A}}(\hat{M}_1^{(e_2)}, \hat{M}_1^{(e_1)}) \hookrightarrow \operatorname{Hom}_{\hat{B}}(\hat{B}t^{1/q_1}, \hat{B}t^{1/q_2}) \cong \hat{B}$, and $\hat{\Psi}(\phi) = \sum_{j\geq 0} \beta_j t^j \in \hat{B}$ $(\beta_j \in K)$. Since ϕ is an isomorphism, it follows that that $\beta_0 \neq 0$ and $\beta_0 V_1^{1/q_1} = V_1^{1/q_2}$, which is a contradiction.

Example 3.4. Let $k = \bigcup_{e\geq 1} \mathbb{F}_2(u^{1/2^e})$, $A = k[x,y]/(x^6 + xy^5 + uy^6)$, $\deg x = \deg y = 1$, and $\hat{A} = k[[x,y]]/(x^6 + xy^5 + uy^6)$. Then A does not have finite graded F-representation type, and $\hat{A}$ does not have finite F-representation type.

If k is a finite field, then we can prove that A has finite F-representation type.

Theorem 3.5. *Let A be a one dimensional complete local or $\mathbb{N}$-graded domain of prime characteristic p. If k a finite field, then A has finite F-representation type.*

Proof. In the case where A is an $\mathbb{N}$-graded ring, since $\{V_i^{1/q} \mid q = p^e,\ i \geq 0\}$ is a finite set, we have the assertion by Lemma 3.2.

Assume that $A = (A, \mathfrak{m}, k)$ is a one-dimensional complete local domain. Let $B = K[[t]]$ be a normalization of A, and set $D = k + tB = k[[\alpha t \mid \alpha \in K]]$. For $\sum_{i\geq n} \beta_i t^i \in B$, $\beta_n \neq 0$, we set $\operatorname{in}_B(f) = \beta_n t^n$. For $i \in \mathbb{N}$, let

$$V_i = \{\beta \mid \operatorname{in}_B(f) = \beta t^i \text{ for some } f \in A\}.$$

Since $\dim_k B/A < \infty$, it follows that $V_i = K$ for all sufficiently large i. We set

$$c = \min\{i \mid V_j = K \text{ for all } j \geq i\}.$$

Since A is complete, we have $\beta t^n \in A$ for all $\beta \in K$ and $n \geq c$. Therefore, $D^q \subset A$ for all $q = p^e \geq c$, and thus $A \subset D \subset A^{1/q}$. In particular, $A^{1/q}$ is a D-module. For i with $V_i \neq 0$, there exists finite set $G_i \subset A$ satisfying following properties:

(1) $\{\operatorname{in}_B(g) \mid g \in G_i\}$ is a k-basis of $V_i t^i = \{\beta t^i \mid \beta \in V_i\}$.
(2) For all $g \in G_i$, g has a form $g = \beta_i t^i + \sum_{j>i} \beta_j t^j$, $\beta_j \notin V_j$.

Note that if $i \geq c$ then $G_i = \{\alpha_1 t^i, \ldots, \alpha_r t^i\}$ where $r = [K : k]$ and $\alpha_1, \ldots, \alpha_r$ is a k-basis of K. It is easy to prove that $\bigcup_{i=0}^{q-1} G_i^{1/q}$ is a system of generators of $A^{1/q}$ as

a D-module for $q \geq c$. Let

$$\begin{aligned} N^{(e)} &= D \cdot (\bigcup_{i=0}^{c-1} G_i^{1/q}) \\ M_i^{(e)} &= D \cdot G_i^{1/q} \text{ for } c \leq 0 \leq q-1. \end{aligned}$$

Then

$$A^{1/q} = (\bigoplus_{i=c}^{q-1} M_i^{(e)}) \oplus N^{(e)},$$

and $M_i^{(e)} \cong B$ for all $c \leq i \leq q-1$. Assume that $\#K = p^f$. To complete the proof, we prove that $N^{(e_1)} \cong N^{(e_2)}$ for $e_1, e_2 \in \mathbb{N}$ with $q_1 = p^{e_1}, q_2 = p^{e_2} \geq c$, and $e_1 \equiv e_2 \mod f$. Let $\varphi : \bigoplus_{i=0}^{c-1} Bt^{i/q_1} \to \bigoplus_{i=0}^{c-1} Bt^{i/q_2}$, $t^{i/q_1} \mapsto t^{i/q_2}$ be an isomorphism of free B-modules (and hence an isomorphism as D-modules). Note that $N^{(e_j)}$ is a D-submodule of $\bigoplus_{i=0}^{c-1} Bt^{i/q_j}$ for $j = 1, 2$ by the definition of G_i. Since $\beta^{p^f} = \beta$ for $\beta \in K$, $\varphi(g^{1/q_1}) = g^{1/q_2}$ for $g = \sum_{i=0}^{c-1} \beta_i t^i \in B$ if $e_1 \equiv e_2 \mod f$. Therefore φ induces one-to-one correspondence between $\bigcup_{i=0}^{c-1} G_i^{1/q_1}$ and $\bigcup_{i=0}^{c-1} G_i^{1/q_2}$ if $e_1 \equiv e_2 \mod f$. This implies that if $e_1 \equiv e_2 \mod f$, the restriction of φ to $N^{(e_1)}$ is an isomorphism form $N^{(e_1)}$ to $N^{(e_2)}$ as D-modules, and thus as A-modules. Therefore, A has finite F-representation type. $\square$

4. Higher dimensional case

We end this paper with a few observations on higher dimension rings of finite F-representation type. Let k be a field of positive characteristic p with $[k : k^p] < \infty$. We begin with the question posed by Brenner.

Question 4.1 (Brenner)**.** Does the ring $k[x, y, z]/(x^2 + y^3 + z^7)$ have finite F-representation type?

Observation 4.2. Let S be an F-finite Cohen-Macaulay local (resp. graded) ring of finite (resp. graded) Cohen-Macaulay type, and R a local ring such that $S \subset R \subset S^{1/q'}$ for some $q' = p^{e'}$. Let M be an R-module (resp. a graded R-module). Since $(S^{1/q'})^q \subset R$ for $q \geq q'$, ${}^eM$ has an $S^{1/q'}$-module structure for $e \geq e'$. If M is a maximal Cohen-Macaulay R-module, then ${}^eM$ is a maximal Cohen-Macaulay $S^{1/q'}$-module, and thus M has finite F-representation type. In particular, if R is Cohen-Macaulay, then R has finite F-representation type.

Example 4.3. Let $R = k[s^q, st, t] \cong k[x, y, z]/(y^q - xz^q)$. Since $k[s^q, t^q] \subset R \subset (k[s^q, t^q])^{1/q}$, R has finite F-representation type.

Example 4.4. Let S be an F-finite regular local ring (resp. a polynomial ring over a field), and let $f \in S$ be an element (resp. a homogeneous element), and $R = S[f^{1/q}]$. Then R has finite F-representation type. In particular, $k[x, y, z]/(x^2 + y^3 + z^7)$ has finite F-representation type if $p = 2, 3$, or 7.

We can prove a little more general theorem.

Theorem 4.5. *Let R be an F-pure complete local (resp. graded) domain of finite F-representation type, $e_1, \dots, e_r$ positive integers, and $q_i = p^{e_i}$. Let $f_1, \dots, f_r$ be (resp. homogeneous) elements of R, and*

$$S = R[x_1, \dots, x_r]/(x_1^{q_1} + f_1, \dots, x_r^{q_r} + f_r).$$

Then S has finite (resp. graded) F-representation type.

Proof. Note that if R is a graded ring, then S is also a graded ring by assigning $\deg(x_i) = \deg(f_i)/q_i$.

Let $\tilde{e} = \max\{e_1, \dots, e_r\}$ and $\tilde{q} = p^{\tilde{e}}$. First, we prove the theorem in the case of $f_i = 0$ for all i. Since $S = R[x_1, \dots, x_r]/(x_1^{q_1}, \dots, x_r^{q_r})$ is a free R-module of finite rank, S has finite F-representation type as an R-module. On the other hand, since $(x_1, \dots, x_r) \cdot {}^eS = (x_1, \dots, x_r)^{[q]}S = 0$ for $e \geq \tilde{e}$, a decomposition of ${}^eS$ as an R-module can be regarded a decomposition as an S-module. Hence S has finite F-representation type.

In the general case, since R is F-pure, R is a direct summand of $R^{1/\tilde{q}}$, and thus $R[x_1, \dots, x_r]$ is a direct summand of $R^{1/\tilde{q}}[x_1, \dots, x_r]$. Hence S is a direct summand of

$$\begin{aligned} & R^{1/\tilde{q}}[x_1, \dots, x_r]/(x_1^{q_1} + f_1, \dots, x_r^{q_r} + f_r) \\ = \; & R^{1/\tilde{q}}[x_1, \dots, x_r]/((x_1 + f_1^{1/q_1})^{q_1}, \dots, (x_r + f_r^{1/q_r})^{q_r}) \\ \cong \; & R^{1/\tilde{q}}[x_1, \dots, x_r]/(x_1^{q_1}, \dots, x_r^{q_r}). \end{aligned}$$

Since $R^{1/\tilde{q}}$ has finite F-representation type, $R^{1/\tilde{q}}[x_1, \dots, x_r]/(x_1^{q_1}, \dots, x_r^{q_r})$ has finite F-representation type as proved above. Therefore S has finite F-representation type by Example 2.2 (v) and (vi). $\square$

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